

Cylindrically symmetric spinning Brans-Dicke spacetimes with closed timelike curves

Luis A. Anchordoqui, Santiago E. Perez Bergliaffa and Marta L. Trobo

Departamento de Física, Universidad Nacional de La Plata

C.C. 67, (1900) La Plata

Argentina

Graciela S. Birman

Departamento de Matemática, Fac. Cs. Exactas, UNCPBA

Pinto 399, (7000) Tandil

Argentina

Abstract

We present here three new solutions of Brans-Dicke theory for a stationary geometry with cylindrical symmetry in the presence of matter in rigid rotation with $T^\mu_\mu \neq 0$. All the solutions have eternal closed timelike curves in some region of spacetime the size of which depends on ω . Moreover, two of them do not go over a solution of general relativity in the limit $\omega \rightarrow \infty$.

PACS number(s): 04.50+h, 04.20Jb

General Relativity (GR) has survived many experimental tests during its rather long existence. However, there remain some conceptual difficulties (mostly related to Quantum Gravity) that have led to the construction of several alternative theories (for a review, see [1]). Brans-Dicke theory (BDT) [2] (along with its generalization, namely the family of scalar-tensor theories [3]) seems to be the most promising of these because of its excellent agreement with experiment and its intimate relationship with the low energy limit of string theory [4]. To be more precise, BDT is consistent with solar-system experiments if the coupling parameter satisfies the inequality $|\omega| > 500$ [5]. As a consequence, a great deal of effort has been devoted to explore the whole space of solutions of BDT. We will be concerned here with solutions that display causal anomalies in the form of closed timelike-curves (CTCs). It has been known for a long time that CTCs are a distinctive feature of geometries in which the light cones may be tilt in an appropriate fashion. The most widely known geometries of these type are the so-called wormholes [6] which are exact solutions of a given theory of gravitation with non-trivial topology (see [7] for wormholes in GR, and [8] for the case of BDT). They do not exhibit CTCs *ab initio*; some (rather artificial) procedures to generate them are explained in [9]. On the other hand, the desired tilting is naturally achieved in rotating space-times such as the spinning black hole of Kerr [10], the infinite dust cylinder of van Stockum [11], the universe of Gödel [12], and their generalizations [13]. In this paper we present solutions to BDT for a cylindrically symmetric spacetime with rigidly rotating matter modeled by a stress-energy tensor with nonzero tangential stresses. We shall see that the solutions have eternal closed timelike curves in a certain region of spacetime, the boundary of which depends on the value of ω . We also analyze the singularity structure of the solutions. Finally, we study the $\omega \rightarrow \infty$ limit, and we find that two of our solutions do not go over solutions of GR.

If for simplicity units are chosen which make the velocity of light equal to unity, the field

equations of BDT are ¹ [2]

$$G_{\mu\nu} = \frac{8\pi}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2} (\phi_{;\mu} \phi_{;\nu} - \frac{1}{2} g_{\mu\nu} \phi^{;\alpha} \phi_{;\alpha}) + \frac{1}{\phi} (\phi_{;\mu\nu} - g_{\mu\nu} \phi^{;\alpha}_{;\alpha}), \quad (1)$$

$$\square\phi \equiv \phi^{;\alpha}_{;\alpha} = \frac{8\pi T}{3 + 2\omega} \quad (2)$$

where $G_{\mu\nu}$ is the Einstein tensor, $T_{\mu\nu}$ is the energy-momentum tensor of matter and all nongravitational fields, and $T = T^\mu_\mu$. In a stationary spacetime the metric and the scalar field can be chosen in such a way that $g_{\mu\nu,0} = \phi_{,0} = 0$, and $g_{0i} = 0$. Further simplification is possible if we assume that the geometry has cylindrical symmetry. The line element can be written in this case as follows:

$$ds^2 = dt^2 - e^{2\lambda(r)}(dr^2 + dz^2) - l(r)d\theta^2 + 2m(r)d\theta dt. \quad (3)$$

We shall take for the stress-energy tensor the expression,

$$T_{\mu\nu} = \rho u_\mu u_\nu + \Pi_{\mu\nu} \quad (4)$$

with ρ the density of energy, and $u^\mu = \delta^\mu_0$ the four velocity in the comoving system. $\Pi_{\mu\nu}$ is the anisotropic pressure tensor, which is symmetric, trace-free, and orthogonal to the comoving observer. We will restrict ourselves here to the case in which $\Pi_{\mu\nu} = \text{diag}(0, -\alpha, \alpha, 0)$ ².

With the foregoing assumptions, the nonvanishing field equations are

¹In what follows, Greek indices run from 0 to 3 while Latin indices from 1 to 3. Semicolons denote covariant derivative with respect to the metric $g_{\mu\nu}$, primes derivatives with respect to r , and the commas mean partial derivatives with respect to the coordinate x^μ . As usual, $g \equiv \det g_{\mu\nu}$ and δ^ν_μ is the Kronecker delta. Sub-indices with hat refer to the proper reference frame, in which $g_{\hat{\mu}\hat{\nu}} \equiv \eta_{\mu\nu}$, being $\eta_{\mu\nu}$ the metric of Minkowski spacetime.

²Let us remark that a solution for the same geometry in the case of dust in BDT was found by Bandyopadhyay [14], but it exhibits the undesirable feature of a singularity at a finite radial proper distance from the origin.

$$\frac{d}{dr} \left(\frac{mm'}{2D} \right) = \frac{4\pi}{\phi} \sqrt{-g} \rho - \frac{mm'}{2D} \frac{\phi'}{\phi} + \frac{1}{2} \frac{\square\phi}{\phi} \sqrt{-g} \quad (5a)$$

$$\frac{d}{dr} \left(\frac{l' + mm'}{2D} \right) = -\frac{4\pi}{\phi} \sqrt{-g} \rho - \frac{mm' + l'}{2D} \frac{\phi'}{\phi} + \frac{1}{2} \frac{\square\phi}{\phi} \sqrt{-g} \quad (5b)$$

$$\frac{d}{dr} \left(\frac{m'}{2D} \right) = -\frac{m'}{2D} \frac{\phi'}{\phi} \quad (5c)$$

$$\frac{d}{dr} \left(\frac{ml' - m'l}{2D} \right) = -\frac{8\pi}{\phi} \sqrt{-g} m \rho + \frac{m'l - ml'}{2D} \frac{\phi'}{\phi} \quad (5d)$$

$$-D\lambda'' + \frac{m'^2}{2D} + D'\lambda' - D'' = \frac{4\pi}{\phi} \sqrt{-g} (\rho - 2\alpha) + \omega D \frac{\phi'^2}{\phi^2} + D \frac{\phi''}{\phi} - D\lambda' \frac{\phi'}{\phi} - \frac{1}{2} \frac{\square\phi}{\phi} \sqrt{-g} \quad (5e)$$

$$-D\lambda'' - D'\lambda' = \frac{4\pi}{\phi} \sqrt{-g} (\rho + 2\alpha) + D\lambda' \frac{\phi'}{\phi} - \frac{1}{2} \frac{\square\phi}{\phi} \sqrt{-g} \quad (5f)$$

$$\square\phi = \frac{8\pi}{3 + 2\omega} \rho \quad (5g)$$

$$D^2 = l + m^2. \quad (5h)$$

To solve this system, first note that Eq. (5c) readily gives the first integral

$$\frac{m'}{2D} = \frac{b}{\phi} \quad (6)$$

where b is an integration constant. From Eqs. (5a) and (5g) we obtain at once that

$$\frac{2D b^2}{\phi} = (2 + \omega) \sqrt{-g} \square\phi \quad (7)$$

whereas from the definition of $\square\phi$ one has

$$D = \exp \left\{ - \int \frac{2b^2 + (2 + \omega)\phi\phi''}{(2 + \omega)\phi\phi'} dr \right\} \quad (8)$$

Eq. (5d) in conjunction with Eqs. (5h), (6) and (8) determine a differential equation for the BD scalar. We shall propose three different types of solutions. In the case of Type I

solutions we assume that the scalar field is given by a polynomical expression in r ; in Type II, by a constant times a trigonometric function, and in Type III we assume that ϕ is given by a constant times an hyperbolic function. After an appropriate fixing of the constants to recover GR whenever it is possible, we obtain the expressions of $\phi(r)$, $l(r)$ and $m(r)$ listed in Table I, along with the range of values of ω in which each solution is valid. Finally, adding Eqs. (5e) and (5f) we obtain an equation for λ ,

$$\lambda'' = \frac{\sqrt{-g}(\omega + 1)}{D} \frac{\square\phi}{\phi} - \frac{\omega}{2} \frac{\phi'^2}{\phi^2} - \frac{1}{2} \frac{\phi''}{\phi} + \frac{m'^2}{4D^2} - \frac{D''}{2D} \quad (9)$$

Integrating this we obtain the expressions listed in Table II.

In order to analyze the singularity structure of these spacetimes, we next study the behaviour of certain scalars built from the Riemann tensor. It is easier to compute this tensor in the orthonormal frame. Let us define the differential forms

$$\Theta^0 = -m d\theta + dt \quad \Theta^1 = e^\lambda dr \quad \Theta^2 = e^\lambda dz \quad \Theta^3 = D d\theta \quad (10)$$

with the corresponding basis vectors

$$e_{\hat{0}}^i = \delta_0^i \quad e_{\hat{1}}^i = e^{-\lambda} \delta_1^i \quad e_{\hat{2}}^i = e^{-\lambda} \delta_2^i \quad e_{\hat{3}}^i = D^{-1} \delta_3^i + m D^{-1} \delta_0^i. \quad (11)$$

The relationship between the Riemann tensor in the two frames is the usual one:

$$R_{\hat{\mu}\hat{\nu}\hat{\eta}\hat{\delta}} = e_{\hat{\mu}}^\alpha e_{\hat{\nu}}^\beta e_{\hat{\eta}}^\epsilon e_{\hat{\delta}}^\psi R_{\alpha\beta\epsilon\psi}$$

In the following we shall see that the first two solutions of Table I are unphysical. From the expression of $R_{\hat{\mu}\hat{\nu}\hat{\eta}\hat{\delta}}$ for the polynomial case, it can be seen that there is a singularity at the origin unless $\Upsilon = 0$. This implies that $\alpha = 0$, and so the solution reduces to the dust case previously analyzed in [14]³. It is worth recalling that in this case there is a singularity which occurs at a finite proper distance from the axis of symmetry, since

³Actually, one still has to require that spacetime be Euclidean near the origin. This fixes C to zero.

$\int_0^\infty \exp(\lambda) dr$ is convergent. Solutions of Type II has instead naked singularities for $r = n\pi/2$, with $n \in \mathcal{N}$. Consequently, from now on, we shall concentrate only on Type III solutions (valid for $\omega \in (-\infty, -2)$) for which the components of the Riemann tensor are well behaved throughout the entire spacetime.

The result of the calculation of the kinematical quantities associated with the fluid shows that the expansion scalar, the acceleration vector, and the shear tensor are all zero, but there is a nonzero vorticity vector given (in the case of Type III solutions) by

$$\omega^\mu = \left(0, 0, -\sqrt{|\omega|/2 - 1} (\cosh r)^{-3-2\omega} e^{(1+\omega/2)r^2}, 0\right) \quad (12)$$

Let us turn now to the analysis of the relation between cause and effect in our spacetime. First, it is necessary to remark that, although in the metric of Type I the coordinate θ is an angular coordinate, this is not the case in Type III. However, after a trivial coordinate transformation the line element may be written in the form

$$ds^2 = dt^2 - e^{2\Lambda(r)} (dr^2 + dz^2) - L(r) d\vartheta^2 + 2M(r) dt d\vartheta \quad (13)$$

where

$$\Lambda(r) = (1 + \omega) \ln(\cosh r) - (2 + \omega) r^2/4,$$

$$L(r) = F^2(r)[1 + 2(2 + \omega) \sinh^2 r] \operatorname{sech}^2 r,$$

and

$$M(r) = \sqrt{2|\omega| - 4} F(r) \tanh r,$$

where $F(r)$ is any function that goes to 0 as r^2 . The coordinate ϑ is now an angular coordinate. Consequently, any curve with constant t , r , and z is closed. In particular, such closed curves are timelike (but they are not geodesics) if $r > r_{\text{crit}} = \operatorname{arcsinh}[1/(4 + 2\omega)]$.

In order to characterize in more detail the matter that generates the geometry, we determine next whether it violates the weak energy condition [15]⁴ To do so, we calculate $\alpha(r)$ from Eq. (5f)

⁴We stress that we are concerned here only with the matter sector of the stress-energy tensor. Note

$$4\pi\alpha = -b \frac{\sqrt{2(|\omega| - 2)}}{4} (\cosh r)^{2|\omega|-1} \exp \left\{ -\frac{|\omega| - 2}{2} r^2 \right\} \quad (14)$$

Finally from (5g), ρ comes out as

$$4\pi\rho = \frac{b}{\sqrt{2(|\omega| - 2)}} (2|\omega| - 3) (\cosh r)^{2|\omega|-3} \exp \left\{ -\frac{|\omega| - 2}{2} r^2 \right\} \quad (15)$$

It is straightforward to compute that $\rho > 0, \forall r$ if $b > 0$. Thus matter will satisfy WEC if and only if $\rho \pm \alpha > 0$ or equivalently if $r > r_{\text{WEC}} = \text{arcosh} \left[\sqrt{\frac{2(2|\omega|-3)}{|\omega|-2}} \right]$.

From Fig. 1, we can see that the radius of the causal region decreases fast with ω , and that there will be CTCs both in the presence of “exotic” and “non-exotic” matter [6] for all values of ω . It can also be seen that the radius of the region in which matter violates WEC tends rapidly to a constant.

We close with some remarks regarding the limit $\omega \rightarrow \infty$. The study of the limits of geometries depending on some parameter has been initiated by Geroch [16], who pointed out that such limit may depend on the coordinate system chosen to perform the calculations. More recently, Paiva *et al* [17] developed a coordinate-free approach (based on the characterization of a given space-time by the Cartan scalars [18]) to study these limits in GR. Romero and Barros found later [19] some examples of BD solutions which do not reduce to GR when $\omega \rightarrow \infty$. This motivated the application of the above mentioned coordinate-free method to BDT by Paiva and Romero [20]. The method involves a lengthy calculation (crucial to make any statement about the limit $\omega \rightarrow \infty$) which will be presented elsewhere. Note however that the BD scalar in solutions of Type II and III goes to zero in the limit $\omega \rightarrow \infty$, and so the effective gravitational constant $G_{\text{eff}} = G_0/\phi$ diverges. Clearly our solutions do not go over GR, and are then counterexamples to the recent claim by Banerjee and Sen that the condition $T \neq 0$ is a necessary and sufficient condition for BD solutions to yield the

however that one should take into account both the BD and the matter sectors of the stress-energy tensor in order to relate violations of WEC to the behaviour of geodesics.

corresponding solutions with the same energy-momentum tensor [21].

ACKNOWLEDGMENTS

Special thanks go to D. Torres for a critical reading of the manuscript. L.A.A. would like to thank FOMEC for financial support. S.E.P.B. was supported by UNLP. M.L.T. and G.S.B were supported by CONICET.

TABLES

TABLE I. The table shows different choices of the BD scalar ϕ , the corresponding metric coefficients, and the maximum range of ω for which each solution exists.

$\phi(r)$	$l(r)$	$m(r)$	Domain of ω
$1 - r^2 b^2 / (4 + 2\omega)$	$(r^2 - b^2 r^4) [1 - r^2 b^2 / (4 + 2\omega)]^{-2}$	$br^2 [1 - r^2 b^2 / (4 + 2\omega)]^{-1}$	$(-\infty, -2)$
$(b\sqrt{2}/\sqrt{2+\omega}) \cos r$	$[1 - 2(2 + \omega) \sin^2 r] \sec^2 r$	$\sqrt{4 + 2\omega} \tan r$	$(-2, \infty)$
$(b\sqrt{-2}/\sqrt{2+\omega}) \cosh r$	$[1 + 2(2 + \omega) \sinh^2 r] \operatorname{sech}^2 r$	$\sqrt{2 \omega - 4} \tanh r$	$(-\infty, -2)$.

TABLE II. The table shows the three different expressions for $\lambda(r)$ in correspondence with those given in Table I. Their behavior with the radial coordinate is also indicated.

$\lambda(r)$	Typical feature of the solution
$(1 + \omega) \ln[2(2 + \omega) - b^2 r^2] + \Upsilon r + \mathcal{C}$	Singularity at a finite proper distance.
$(1 + \omega) \ln(\cos r) + (2 + \omega) r^2 / 4$	Naked singularities at $r = n\pi/2$.
$(1 + \omega) \ln(\cosh r) - (2 + \omega) r^2 / 4$	Well behaved throughout the spacetime.

REFERENCES

- [1] H.F.M. Goenner in *General Relativity and Gravitation*, Proceedings of GR11 Conference, Stockholm 1986. ed. M. MacCallum, Cambridge U. Press (1987).
- [2] C. Brans and R. H. Dicke, Phys. Rev. **124**, 925 (1961).
- [3] J. D. Barrow, Phys. Rev. D **47**, 5329 (1993).
- [4] M. B. Green, J. H. Schwartz, and E. Witten, *Superstring Theory*, Cambridge U. Press (1988).
- [5] C. M. Will, Phys. Rep. **113**, 345 (1984).
- [6] M. Morris and K. Thorne, Am. J. Phys. **56**, 395 (1988).
- [7] M. Visser, *Lorentzian Wormholes*, AIP Press (1996).
- [8] P. S. Letelier and A. Wang, Phys. Rev. D **48**, 631 (1993); A. Agnese and M. LaCamera, Phys. Rev. D **51**, 2011 (1995); K. K. Nandi, A. Islam and J. Evans, Phys. Rev. D **55**, 2497 (1997); L. A. Anchordoqui, S. E. Perez Bergliaffa and D. F. Torres, Phys. Rev. D **55**, 5226 (1997); K. K. Nandi, B. Bhattacharjee, S. M. K. Alam and J. Evans, Phys. Rev. D **57**, 823 (1998).
- [9] M. Morris, K. Thorne and U. Yurtserver, Phys. Rev. Lett. **61**, 1446 (1988); V. Frolov and I. Novikov, Phys. Rev. D **42** 1913 (1990); M. Visser, Phys. Rev. D **55**, 5212 (1997).
- [10] R. P. Kerr, Phys. Rev. Lett. **11**, 237 (1963).
- [11] W. J. van Stockum, Proc. Roy. Soc. Edin. **57**, 135 (1937). For mathematical details see W. B. Bonnor, J. Phys. A: Math. Gen. **13**, 2121 (1980), and for details about the causal structure, see F. J. Tipler, Phys. Rev. D **9**, 2203 (1974).
- [12] K. Gödel, Rev. Mod. Phys. **21**, 447 (1949).
- [13] J. P. Wright, J. Math. Phys. **6**, 103 (1965); M. M. Som and A. K. Raychaudhuri, Proc.

- Roy. Soc. A **304**, 81 (1968); M. Novello and M. J. Reboucas, Phys. Rev. D **19** 2850 (1979); A. K. Raychaudhuri and S. N. Guha Thakurta, Phys. Rev. D **22** (1980); S. K. Chakraborty and N. Bandyopadhyay, J. Math. Phys. **24**, 129 (1983).
- [14] N. Bandyopadhyay, J. Math. Phys. **19**, 1423 (1978).
- [15] S. Hawking and G. Ellis, *The Large Scale Structure of Space Time*, Cambridge U. Press, Cambridge, England (1973).
- [16] R. Geroch, Comm. Math. Phys. **13**, 180 (1969).
- [17] F. Paiva, M. Reboucas and M. MacCallum, Class. Quant. Grav. **10**, 1165 (1993).
- [18] E. Cartan, *Leçons sur la Géométrie des Espaces de Riemann*, Gauthier-Villars, Paris (1951).
- [19] C. Romero and A. Barros, Phys. Lett. A **173**, 243 (1993).
- [20] F. Paiva and C. Romero, Gen. Rel. Grav. **25**, 1305 (1993).
- [21] N. Banerjee and S. Sen, Phys. Rev. D **56**, 1334 (1997).

FIGURES

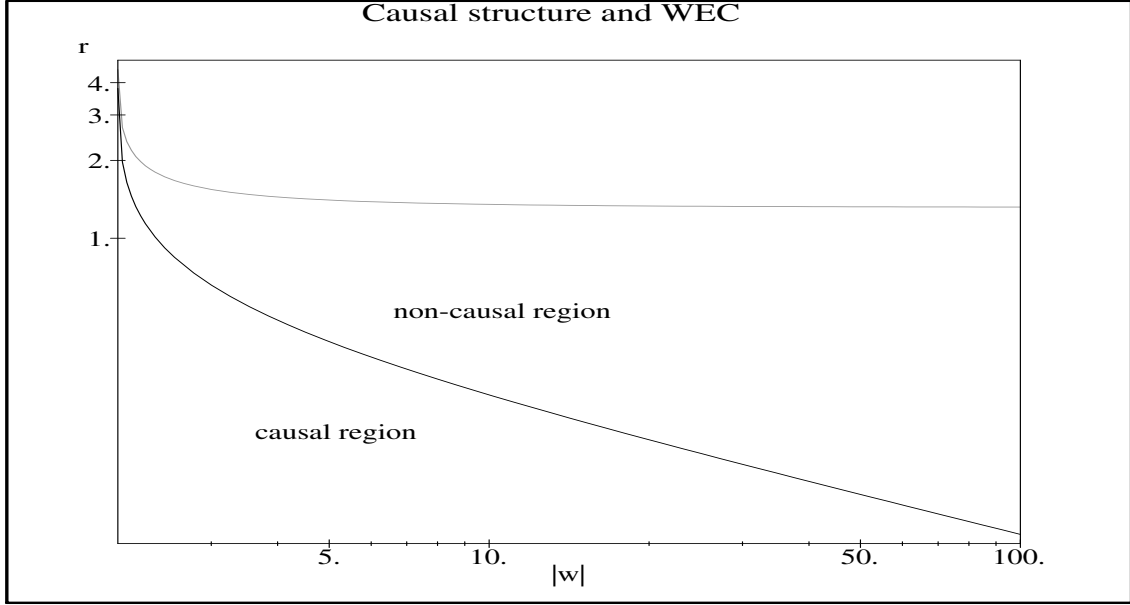


FIG. 1. Variation of r_{crit} (black line) and r_{WEC} (grey line) with $|\omega|$.